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ON A NESTED BOUNDARY-LAYER PROBLEM

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Dedicated to Professor Philippe G. Ciarlet on the occasion of his 70th birthday

Abstract. Nested boundary layers mean that one boundary layer lies inside another one. In this paper, we consider one such problem, namely,

$$\begin{cases} \varepsilon^3 xy''(x) + x^2 y'(x) - (x^3+\varepsilon)y(x) = 0, & 0 < x < 1, \\ y(0) = 1, \qquad y(1) = \sqrt{e}. \end{cases}$$

An asymptotic solution, which holds uniformly for $x \in [0,1]$, is constructed rigorously. This result also provides an explicit formula for the exponentially small leading term in the interval where the exact solution exhibits such behavior. This phenomenon has never been mentioned in the existing literature.

1. **Introduction.** An interesting problem in singular perturbation theory is to consider the *nested boundary layers*, which mean that one boundary layer lies inside another one. A well-known boundary value problem with such a phenomenon is given by

$$\varepsilon^3 xy''(x) + x^2 y'(x) - (x^3+\varepsilon)y(x) = 0 \tag{1.1}$$

with boundary conditions

$$y(0) = 1, \qquad y(1) = \sqrt{e}; \tag{1.2}$$

see Bender and Orszag [2, p.453]. By using the method of matched asymptotics, the authors of [2] constructed the one-term asymptotic solution

$$y_{\text{unif}} = \frac{2\sqrt{x}}{\varepsilon} K_1\left(\frac{2\sqrt{x}}{\varepsilon}\right) + e^{-\frac{\varepsilon}{x}} + e^{\frac{x^2}{2}} - 1, \tag{1.3}$$

where the first three terms represent the leading terms in the expansions of the *inner-inner solution, inner solution and outer solution*, respectively, and the fourth term results from the matching of the inner solution and outer solution.

The two boundary layers arise from the two corresponding distinguished limits ε and ε^2. Distinguished limits are those proper stretchings which produce a nontrivial balance between two or more terms of the equation. To determine the distinguished limits, we let

$$y(x) = Y(X) \qquad \text{and} \qquad X = x/\delta.$$

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With these new variables, (1.1) becomes

$$\frac{\varepsilon^3}{\delta} X \frac{d^2Y}{dX^2} + \delta X^2 \frac{dY}{dX} - \delta^3 X^3 Y - \varepsilon Y = 0. \tag{1.4}$$

For the scaling $\delta = \varepsilon$, the second and fourth terms of (1.4) are of comparable size while the first and third are smaller. Regarding the other choice $\delta = \varepsilon^2$, it is clear that the first and fourth terms are of same order while the rest are smaller.

As we all know, for single boundary layer problems, the inner solution matches the outer solution and also satisfies one boundary condition. Unfortunately, the inner solution for the boundary value problem (1.1) − (1.2)

$$Y_{in}(X;\varepsilon) = e^{-1/X}\left[1 + \varepsilon\left(1 - \frac{2}{3X^3} + \frac{1}{4X^4}\right)\right] + \cdots, \tag{1.5}$$

where $X = x/\varepsilon$, does not satisfy the boundary condition $y(0){=}1$ since $Y_{in}(X;\varepsilon)$ vanishes exponentially fast as $X \to 0^+$. Intuition suggests that an additional boundary layer very near $x = 0$ might be required in order to satisfy the condition $y(0) = 1$. Since $\delta = \varepsilon^2$ is the only other distinguished limit for (1.1), Bender and Orszag obtained the inner-inner solution

$$\bar{Y}(\bar{X};\varepsilon) = 2\sqrt{\bar{X}}K_1(2\sqrt{\bar{X}}) + \cdots, \tag{1.6}$$

where $\bar{X} = x/\varepsilon^2$. Combining (1.5) and (1.6) with the outer solution, one obtains the uniformly valid one-term asymptotic solution (1.3).

However, in lieu of matching the inner-inner solution $\bar{Y}(\bar{X};\varepsilon)$ with the inner solution $Y_{in}(X,\varepsilon)$, Bender and Orszag claimed that the two solutions match automatically because they both vanish exponentially. Indeed, they are exponentially small, but they may not be of the same order. Thus, the solution (1.3) is only valid up to $O(\varepsilon)$ and cannot give the correct leading term in the matching region where the exact solution may be exponentially small.

The treatment of the nested boundary layers problem is based on Prandtl's principle of matched asymptotics. However, the lack of rigor in Prandtl's boundary layer theory does raise concern from mathematicians who believe that arguments based on purely heuristic reasoning may lead to incorrect results. Examples of this nature can be found in [7, pp. 239-245] and [9]. With the same purpose, we are going to reinvestigate the problem (1.1) − (1.2) and to derive a uniformly valid asymptotic solution by a mathematically rigorous argument. Moreover, we can find explicitly the exponentially small leading term in the interval where the exact solution exhibits such behavior.

Before proceeding, let us look at one more example: a triple-deck problem. For a nested boundary-layer problem, the domains of validity of the outer solution, inner solution and inner-inner solution are called outer region, inner region and inner-inner region, accordingly. Furthermore, these layers are also known as decks, especially by physicists; in other words, they are called right deck, middle deck and left deck, respectively. Hence, a nested boundary-layer problem is also referred to as a triple-deck problem. Several typical examples are given in [4] and [5]. In [4, pp. 304-307], Nayfeh considered the boundary-value problem

$$\begin{cases} \varepsilon^3 y''(x) + x^3 y'(x) + (x^3 - \varepsilon)y(x) = 0 \\ y(0) = \alpha, \qquad y(1) = \beta, \end{cases} \tag{1.7}$$

and gave the composite expansion

$$y^c(x) = \beta e^{1-x} + \beta e^{1-\varepsilon/2x^2} + \alpha e^{-x/\varepsilon} - \beta e + \cdots . \tag{1.8}$$

To illustrate that $y^c(x)$ is close to the exact solution, Nayfeh presented a figure in [4, p.307] to compare $y^c(x)$ with the numerical solution. In Figure 1, we plot the numerical solution, Nayfeh's solution $y^c(x)$ and our rigorously established approximate solution $Y(x)$ given in (6.8).

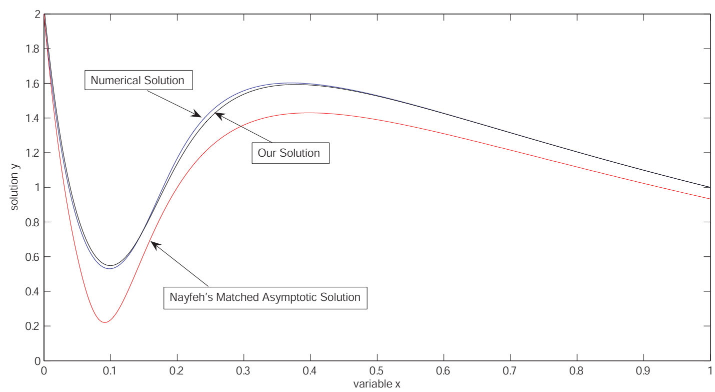

FIGURE 1. Comparison of the composite solution $y^c(x)$ and the asymptotic solution $Y(x)$ with the exact numerical solution when $\alpha = 2.0$, $\beta = 1.0$ and $\varepsilon = 0.05$.

2. **Preliminary transformations.** For the general singularly perturbed two-point boundary value problem (B.V.P.)

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0 \tag{2.1}$$

with the boundary conditions

$$y(0) = A \qquad \text{and} \qquad y(1) = B, \tag{2.2}$$

it is now well-known that if $a(x)$ is positive, then the asymptotic solution which holds uniformly in the interval $[0, 1]$ is given by

$$y_{\text{unif}}(x) = B\exp\left(\int_x^1 \frac{b(t)}{a(t)}dt\right) + \left\{A - B\exp(\int_0^1 \frac{b(t)}{a(t)}dt)\right\} e^{-a(0)\frac{x}{\varepsilon}}. \tag{2.3}$$

This formula can be easily derived by using Prandtl's boundary layer theory; see, e.g., [2, p.425]. Moreover, y_{unif} in (2.3) approximates the true solution of (2.1) in the sense that

$$y(x) = y_{\text{unif}}(x) + O(\varepsilon),$$

where the O-term is uniform with respect to $x \in [0, 1]$; cf. [2, p.479]. As we have mentioned before, heuristic reasoning may lead to incorrect results. For instance, if the boundary value B in (2.2) is zero, then (2.3) becomes

$$y_{\text{unif}}(x) = Ae^{-a(0)\frac{x}{\varepsilon}},$$

which is exponentially small for $x > 0$ and asymptotically equal to zero with respect to the error estimate $O(\varepsilon)$. A more accurate formula for the B.V.P. (2.1) − (2.2) is

$$\begin{aligned} y(x) &= B\exp\left(\int_x^1 \frac{b(t)}{a(t)}dt\right)[1+O(\varepsilon)] \\ &\quad + \frac{a(0)}{a(x)}\left\{A - B\exp(\int_0^1 \frac{b(t)}{a(t)}dt)\right\}\exp\left(\int_0^x \frac{b(t)}{a(t)}dt\right) \\ &\quad \times \exp\left(-\frac{1}{\varepsilon}\int_0^x a(t)dt\right)[1+O(\varepsilon)]. \end{aligned}$$

One can establish this result by using the Liouville-Green (WKB) approximation, which provides good asymptotic solutions to differential equations of the type

$$y''(x) + (\lambda^2 p(x) + q(x))y(x) = 0, \tag{2.4}$$

where λ is a positive large parameter and $p(x)$ is positive. The basic idea in obtaining an asymptotic solution to (2.4) is to introduce the Liouville transformation

$$\xi = \int p^{1/2}(x)dx \qquad \text{and} \qquad w = p^{1/4}(x)y(x). \tag{2.5}$$

When the coefficient function $p(x)$ in (2.4) has a zero (turning point), say $x = x_0$, then there is an ambiguity in taking the square root of the function $p(x)$, and hence the Liouville transformation (2.5) is not well defined. However, a good approximation can be obtained by introducing the Langer transformation [3], which can also be used to obtain asymptotic solutions to internal boundary-layer problems (e.g., when $a(x)$ in (2.1) has a zero in the interval $(0, 1)$; see [8]).

A third situation arises when $p(x)$ in (2.4) has a simple pole at x_0. Again, the transformation for this case was first introduced by Langer, extended and popularized by Olver; see [6]. Our treatment of (1.1) is based on this transformation. The key idea behind this transformation is to get a good approximate equation for (1.1), and to estimate the error term by using the method of successive approximation.

Motivated by the *inner-inner solution* in (1.3), we will try to derive an equation which is a perturbation of Weber's equation for modified Bessel functions. As we all know, the ordinary differential equation

$$\frac{d^2w}{dz^2} = \frac{1}{4z}w \tag{2.6}$$

has two linearly independent solutions

$$\sqrt{z}K_1(\sqrt{z}) \qquad \text{and} \qquad \sqrt{z}I_1(\sqrt{z}).$$

Hence, we will seek a transformation which converts (1.1) into a perturbed equation of (2.6). To this end, we introduce new independent and dependent variables

$$\zeta = \zeta(x) \qquad \text{and} \qquad y(x) = A(x)U(\zeta(x)). \tag{2.7}$$

Straightforward calculation gives

$$\frac{dy}{dx} = A'(x)U(\zeta(x)) + A(x)\frac{dU}{d\zeta}\frac{d\zeta}{dx}$$

and

$$\frac{d^2y}{dx^2} = A''(x)U(\zeta) + 2A'(x)\frac{dU}{d\zeta}\frac{d\zeta}{dx} + A(x)\left(\frac{dU}{d\zeta}\frac{d^2\zeta}{dx^2} + \frac{d^2U}{d\zeta^2}\left(\frac{d\zeta}{dx}\right)^2\right).$$

With the new variables, (1.1) becomes

$$\begin{aligned}\varepsilon^3 A(x)\left(\frac{d\zeta}{dx}\right)^2\frac{d^2U}{d\zeta^2}+\left(2\varepsilon^3A'(x)\frac{d\zeta}{dx}+\varepsilon^3A(x)\frac{d^2\zeta}{dx^2}+xA(x)\frac{d\zeta}{dx}\right)\frac{dU}{d\zeta}\\+\left(\varepsilon^3A''(x)+xA'(x)-(x^2+\frac{\varepsilon}{x})A(x)\right)U=0.\end{aligned}\tag{2.8}$$

To eliminate the first order derivative of $U(\zeta)$, we set

$$2\varepsilon^3A'(x)\frac{d\zeta}{dx}+\varepsilon^3A(x)\frac{d^2\zeta}{dx^2}+xA(x)\frac{d\zeta}{dx}=0.\tag{2.9}$$

Upon solving (2.9), we have

$$A(x)=(\zeta'(x))^{-\frac{1}{2}}\exp\left\{-\frac{x^2}{4\varepsilon^3}\right\}.\tag{2.10}$$

Substituting (2.10) into (2.8) yields

$$\begin{aligned}\frac{d^2U}{d\zeta^2}-\frac{U}{4\zeta}=\left\{\frac{1}{\varepsilon^3}\frac{1}{(\zeta'(x))^2}\left(\frac{x^2}{4\varepsilon^3}+x^2+\frac{1}{2}+\frac{\varepsilon}{x}\right)-\frac{1}{4\zeta}\right\}U\\+\left\{\frac{\zeta(x)}{(\zeta'(x))^3}\left(2\zeta'''(x)-\frac{3(\zeta''(x))^2}{\zeta'(x)}\right)\right\}\frac{U}{4\zeta}.\end{aligned}\tag{2.11}$$

Again, set

$$\frac{1}{\varepsilon^3}\frac{1}{(\zeta'(x))^2}\left(\frac{x^2}{4\varepsilon^3}+x^2+\frac{1}{2}+\frac{\varepsilon}{x}\right)-\frac{1}{4\zeta}=0.\tag{2.12}$$

Solving (2.12), we obtain

$$\zeta^{\frac{1}{2}}(x)=\frac{1}{\varepsilon}\int_0^x\sqrt{\frac{1}{t}+\frac{t^2}{4\varepsilon^4}+\frac{1}{2\varepsilon}(1+2t^2)}\;dt.\tag{2.13}$$

As a consequence, $\zeta=\zeta(x)$ is a well-defined and one-to-one mapping from $[0,1]$ into $[\zeta_-,\zeta_+]$, where

$$\zeta_-=\zeta(0)=0,\quad\text{and}\quad\zeta_+=\zeta(1).$$

Let us summarize the result in the following lemma.

Lemma 1. Under the transformations (2.7), (2.10) and (2.13), (1.1) is converted into

$$\frac{d^2U}{d\zeta^2}-\frac{U}{4\zeta}=\phi(\zeta)\frac{U}{4\zeta},\tag{2.14}$$

where

$$\phi(\zeta)=\frac{\zeta(x)}{(\zeta'(x))^3}\left(2\zeta'''(x)-\frac{3(\zeta''(x))^2}{\zeta'(x)}\right).\tag{2.15}$$

3. **Conversion to an integral equation.** We construct two linearly independent solutions $W_1(\zeta)$ and $W_2(\zeta)$ to the associated homogeneous equation of (2.14), namely,

$$\frac{d^2U}{d\zeta^2}-\frac{U}{4\zeta}=0,\tag{3.1}$$

such that

$$\begin{aligned}W_1(\zeta_-)=1,\quad W_1(\zeta_+)=0,\\W_2(\zeta_-)=0,\quad W_2(\zeta_+)=1.\end{aligned}$$

By straightforward calculation, we have

$$W_1(\zeta)=\sqrt{\zeta}K_1(\sqrt{\zeta})-P\sqrt{\zeta}I_1(\sqrt{\zeta}),$$

$$W_2(\zeta) = \frac{\sqrt{\zeta} I_1(\sqrt{\zeta})}{Q},$$

where

$$P = \frac{K_1(\sqrt{\zeta_+})}{I_1(\sqrt{\zeta_+})}, \quad Q = \sqrt{\zeta_+} I_1(\sqrt{\zeta_+}).$$

Moreover,

$$\mathcal{W}\{W_1(\zeta), W_2(\zeta)\} = \frac{1}{2Q}.$$

By considering the right-hand side of (2.14) as an inhomogeneous term of the homogeneous equation (3.1), we can convert (2.14) into the integral equation

$$U(\zeta) = \lambda W_1(\zeta) + \mu W_2(\zeta) + \int_{\zeta_-}^{\zeta_+} G(\zeta, s)\phi(s)\frac{U(s)}{4s} ds, \tag{3.2}$$

where λ, μ are arbitrary constants, and $G(\zeta, s)$ is the Green's function defined by

$$G(\zeta, s) = \begin{cases} \dfrac{W_1(\zeta)W_2(s)}{\mathcal{W}\{W_1(s), W_2(s)\}} = 2QW_1(\zeta)W_2(s), & \zeta_- \le s \le \zeta \le \zeta_+, \\ \dfrac{W_1(s)W_2(\zeta)}{\mathcal{W}\{W_1(s), W_2(s)\}} = 2QW_1(s)W_2(\zeta), & \zeta_- \le \zeta \le s \le \zeta_+. \end{cases} \tag{3.3}$$

4. **Construction of the Solution.**

Lemma 2. Let ρ be a positive constant. We have the order estimates

$$\phi(x) = O(\frac{x^2}{\varepsilon^2}) \qquad \text{for} \qquad 0 \le x \le \rho\varepsilon^{4/3}, \tag{4.1}$$

$$\phi(x) = O(\frac{\varepsilon^6}{x^4}) \qquad \text{for} \qquad \rho\varepsilon^{4/3} \le x \le 1. \tag{4.2}$$

Proof. For convenience, we put

$$f(t) = \sqrt{\frac{1}{t} + \frac{t^2}{4\varepsilon^4} + \frac{1}{2\varepsilon}(1 + 2t^2)} \qquad \text{for} \quad 0 < t \le 1. \tag{4.3}$$

From (2.13), (2.15) and (4.3), it follows that

$$\phi(\zeta) = \phi_1(x) + \phi_2(x),$$

where

$$\phi_1(x) = -\frac{3}{4}\frac{1}{\zeta(x)} \tag{4.4}$$

and

$$\begin{aligned} \phi_2(x) = \varepsilon^2 f^{-6}(x)\bigg\{ &\frac{3}{16}\frac{1}{x^4} + \frac{1}{4\varepsilon}\frac{1}{x^3} + \frac{9}{4x}\left(\frac{1}{4\varepsilon^4} + \frac{1}{\varepsilon}\right) \\ &+ \frac{1}{4\varepsilon}\left(\frac{1}{4\varepsilon^4} + \frac{1}{\varepsilon}\right) - \frac{3x^2}{4}\left(\frac{1}{4\varepsilon^4} + \frac{1}{\varepsilon}\right)^2 \bigg\}. \end{aligned} \tag{4.5}$$

Let us first consider the case $0 < t \le x \le \rho\varepsilon^{4/3}$. Assume temporarily that x is much smaller than $\varepsilon^{4/3}$ (i.e., $x \ll \varepsilon^{4/3}$), so that

$$f(t) = \frac{1}{\sqrt{t}}\left\{1 + \frac{1}{2}(\frac{t^3}{4\varepsilon^4} + \frac{t}{2\varepsilon})[1 + O(\frac{t^3}{\varepsilon^4}, \frac{t}{\varepsilon})]\right\}, \tag{4.6}$$

$$\zeta^{\frac{1}{2}}(x) = \frac{2\sqrt{x}}{\varepsilon}\left\{1 + (\frac{x^3}{56\varepsilon^4} + \frac{x}{12\varepsilon})[1 + O(\frac{x^3}{\varepsilon^4}, \frac{x}{\varepsilon})]\right\}, \tag{4.7}$$

and

$$\phi_1(x) = -\frac{3}{16}\frac{\varepsilon^2}{x} + (\frac{3x^2}{448\varepsilon^2} + \frac{\varepsilon}{32})[1 + O(\frac{x^3}{\varepsilon^4}, \frac{x}{\varepsilon})], \tag{4.8}$$

where $O(h,k)$ is used to indicate that the error term is of the order $O(h)+O(k)$. Coupling (4.5) and (4.6) yields

$$\phi_2(x) = \frac{3\varepsilon^2}{16x} + \left(\frac{27x^2}{64\varepsilon^2} - \frac{\varepsilon}{32}\right)[1 + O(\frac{x^3}{\varepsilon^4}, \frac{x}{\varepsilon}, \varepsilon^3)]. \tag{4.9}$$

Hence,

$$\phi(x) = \phi_1(x) + \phi_2(x) = \frac{3}{7}\frac{x^2}{\varepsilon^2}[1 + O(\frac{x^3}{\varepsilon^4}, \frac{x}{2\varepsilon}, \varepsilon^3)] = O(\frac{x^2}{\varepsilon^2}). \tag{4.10}$$

Since we are only interested in the leading order estimate for $\phi(x)$, our earlier assumption "$x \ll \varepsilon^{4/3}$" can now be dropped because x^3/ε^4 is still bounded.

For the second case $\rho\varepsilon^{4/3} \le x \le 1$, (2.13) and (4.3) imply that

$$\zeta^{1/2}(x) = \frac{1}{\varepsilon}\int_0^x f(t)dt \ge \frac{1}{\varepsilon}\int_0^x \frac{t}{2\varepsilon^2}dt = \frac{x^2}{4\varepsilon^3}.$$

Thus, we have

$$|\phi_1(x)| = \left|-\frac{3}{4}\frac{1}{\zeta(x)}\right| \le \frac{12\varepsilon^6}{x^4} = O(\frac{\varepsilon^6}{x^4}).$$

In a similar manner, it can be shown that $\phi_2(x) = O(\varepsilon^6/x^4)$. □

Lemma 2 implies that $\phi(x)$ is much smaller than 1, thus we can use the solution of the homogeneous equation (3.1) to approximate the solution of the inhomogeneous equation (2.14). In the following lemma, we are going to estimate the error term caused by discarding the inhomogeneous term of (2.14).

Lemma 3. As $\varepsilon \to 0$, we have

$$\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds = W_1(\zeta)O(\varepsilon^{1/3}), \tag{4.11}$$

$$\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_2(s)ds = W_2(\zeta)O(\varepsilon^{1/3}). \tag{4.12}$$

Proof. Define

$$M_1(\zeta,\varepsilon) := \int_{\zeta_-}^{\zeta} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds,$$

$$M_2(\zeta,\varepsilon) := \int_{\zeta}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds,$$

$$N_1(\zeta,\varepsilon) := \int_{\zeta_-}^{\zeta} G(\zeta,s)\frac{|\phi(s)|}{4s}W_2(s)ds,$$

$$N_2(\zeta,\varepsilon) := \int_{\zeta}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_2(s)ds.$$

First of all, we recall some asymptotic formulas of $K_1(z)$ and $I_1(z)$:

$$K_1(z) \sim \frac{1}{z} \qquad \text{as } z \to 0, \tag{4.13}$$

$$I_1(z) \sim \frac{z}{2} \qquad \text{as } z \to 0, \tag{4.14}$$

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \qquad \text{as } z \to \infty, \tag{4.15}$$

$$I_1(z) \sim \frac{e^z}{\sqrt{2\pi z}} \qquad \text{as } z \to \infty. \tag{4.16}$$

By making the change of variable given in (2.13) with z and s replacing x and ζ, we obtain

$$\begin{aligned} M_1(\zeta,\varepsilon) = O(1)W_1(\zeta) \int_0^x I_1(\sqrt{s(z)})K_1(\sqrt{s(z)})\,|\phi(s(z))| \\ \times 2\sqrt{s(z)}\frac{1}{\varepsilon}\sqrt{\frac{1}{z}+\frac{z^2}{4\varepsilon^4}+\frac{1}{2\varepsilon}(1+2z^2)}dz. \end{aligned} \tag{4.17}$$

To proceed further, we divide our discussion into three cases. Let k_1, k_2 be positive constants.

Case I: $0 < x \le k_1\varepsilon^2$.

For $0 < z < x$, we have from (4.6), (4.7) and (4.10)

$$f(z) = \frac{1}{\sqrt{z}}[1 + O(\frac{z^3}{\varepsilon^4}, \frac{z}{\varepsilon})], \tag{4.18}$$

$$s^{1/2}(z) = \frac{2\sqrt{z}}{\varepsilon}[1 + O(\frac{z^3}{\varepsilon^4}, \frac{z}{\varepsilon})], \tag{4.19}$$

$$\phi(z) = \frac{3z^2}{7\varepsilon^2}[1 + O(\frac{z^3}{\varepsilon^4}, \frac{z}{\varepsilon}, \varepsilon^3)]. \tag{4.20}$$

A combination of (4.13), (4.14), (4.17), (4.18), (4.19) and (4.20) gives

$$M_1(\zeta,\varepsilon) = O(1)W_1(\zeta)\int_0^x \frac{\sqrt{s}}{2}\frac{1}{\sqrt{s}}\frac{3}{7}\frac{z^2}{\varepsilon^2}\frac{4\sqrt{z}}{\varepsilon}\frac{1}{\varepsilon}\frac{1}{\sqrt{z}}dz = W_1(\zeta)O(\varepsilon^2). \tag{4.21}$$

Case II: $k_1\varepsilon^2 \le x \le k_2\varepsilon^{\frac{4}{3}}$.

On account of (4.15), (4.16) (4.19), (4.20) and (4.21), we have

$$\begin{aligned} M_1(\zeta,\varepsilon) &= W_1(\zeta)O(\varepsilon^2) + O(1)W_1(\zeta)\int_{k_1\varepsilon^2}^x \frac{e^{\sqrt{s}}}{\sqrt{2\pi}s^{\frac14}}\sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{s}}}{s^{\frac14}}\frac{3}{7}\frac{z^2}{\varepsilon^2}2\sqrt{s}\frac{1}{\varepsilon}\frac{1}{\sqrt{z}}dz \\ &= W_1(\zeta)O(\varepsilon^{\frac13}). \end{aligned} \tag{4.22}$$

Case III: $k_2\varepsilon^{4/3} \le x \le 1$.

In a similar manner, we obtain

$$\begin{aligned} M_1(\zeta,\varepsilon) &= W_1(\zeta)O(\varepsilon^{\frac13}) + O(1)W_1(\zeta)\int_{k_2\varepsilon^{\frac43}}^x \frac{e^{\sqrt{s}}}{\sqrt{2\pi}s^{\frac14}}\sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{s}}}{s^{\frac14}}\frac{\varepsilon^6}{z^4}2\sqrt{s}\frac{1}{\varepsilon}\frac{z}{2\varepsilon^2}dz \\ &= W_1(\zeta)O(\varepsilon^{\frac13}). \end{aligned} \tag{4.23}$$

Therefore,

$$M_1(\zeta,\varepsilon) = W_1(\zeta)O(\varepsilon^{\frac13}) \qquad \text{for} \qquad \zeta_- \le \zeta \le \zeta_+.$$

For real positive z, $K_1(z)$ is a monotonically decreasing function, while $I_1(z)$ is a strictly increasing function. Making use of this property, we get

$$\begin{aligned} M_2(\zeta,\varepsilon) &= \frac{1}{2}W_1(\zeta)\int_\zeta^{\zeta+} I_1(\sqrt{\zeta})\,|\phi(s)|\,\frac{[K_1(\sqrt{s})-PI_1(\sqrt{s})]^2}{(K_1(\sqrt{\zeta})-PI_1(\sqrt{\zeta}))}ds \\ &\le \frac{1}{2}W_1(\zeta)\int_\zeta^{\zeta+} I_1(\sqrt{s})\,|\phi(s)|\,(K_1(\sqrt{s})-PI_1(\sqrt{s}))ds \\ &= W_1(\zeta)O(\varepsilon^{\frac{1}{3}}). \end{aligned}$$

Hence, as $\varepsilon \to 0$,

$$\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds = M_1(\zeta,\varepsilon)+M_2(\zeta,\varepsilon) = W_1(\zeta)O(\varepsilon^{1/3}) \tag{4.24}$$

uniformly for all $\zeta \in [\zeta_-,\zeta_+]$.

Similar arguments show that

$$N_1(\zeta,\varepsilon) = W_2(\zeta)O(\varepsilon^{\frac{1}{3}}) \quad \text{and} \quad N_2(\zeta,\varepsilon) = W_2(\zeta)O(\varepsilon^{\frac{1}{3}}).$$

Thus, (4.12) follows. □

Remark. Lemma 3 infers that there exist positive constants ε_0 and γ_0 such that for $0 < \varepsilon \le \varepsilon_0$,

$$\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds \le W_1(\zeta)\gamma_0\varepsilon^{1/3} \tag{4.25}$$

and

$$\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_2(s)ds \le W_2(\zeta)\gamma_0\varepsilon^{1/3}. \tag{4.26}$$

Theorem 1. There exists a constant $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, equation (2.14) has two linearly independent solutions

$$w_1(\zeta) = \sqrt{\zeta}K_1(\sqrt{\zeta})(1+O(\varepsilon^{\frac{1}{3}})) \tag{4.27}$$

and

$$w_2(\zeta) = \sqrt{\zeta}I_1(\sqrt{\zeta})(1+O(\varepsilon^{\frac{1}{3}})), \tag{4.28}$$

where the O-symbols hold uniformly with respect to $\zeta \in [\zeta_-,\zeta_+]$.

Proof. In (3.2), we let $\lambda = 1$ and $\mu = 0$. Then

$$U(\zeta) = W_1(\zeta) + \int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{\phi(s)}{4s}U(s)ds. \tag{4.29}$$

We define a sequence $U_j(\zeta)$, $j = 0, 1, ...,$ by $U_0(\zeta) = 0$ and

$$U_j(\zeta) = W_1(\zeta) + \int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{\phi(s)}{4s}U_{j-1}(s)ds \qquad (j \ge 1). \tag{4.30}$$

Clearly, $U_1(\zeta) = W_1(\zeta)$. Now suppose that for a particular value j, we have

$$|U_j(\zeta) - U_{j-1}(\zeta)| \le W_1(\zeta)(\gamma_0\varepsilon^{\frac{1}{3}})^{j-1}, \tag{4.31}$$

as indeed is the case when $j = 1$. From (4.30), we get

$$U_{j+1}(\zeta) - U_j(\zeta) = \int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{\phi(s)}{4s}\left(U_j(s) - U_{j-1}(s)\right)ds \qquad (j \ge 1). \tag{4.32}$$

Hence, (4.25) and (4.32) give

$$\begin{aligned}|U_{j+1}(\zeta)-U_j(\zeta)| &\le \int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)(\gamma_0\varepsilon^{\frac{1}{3}})^{j-1}ds\\ &= (\gamma_0\varepsilon^{\frac{1}{3}})^{j-1}\int_{\zeta_-}^{\zeta_+} G(\zeta,s)\frac{|\phi(s)|}{4s}W_1(s)ds\\ &\le W_1(\zeta)(\gamma_0\varepsilon^{\frac{1}{3}})^{j}.\end{aligned} \tag{4.33}$$

By induction, (4.31) holds for all $j \ge 1$, and the series

$$\widetilde{U_1}(\zeta) := \sum_{j=0}^{\infty}[U_{j+1}(\zeta)-U_j(\zeta)]$$

converges. From (4.33), it follows that

$$\begin{aligned}\widetilde{U_1}(\zeta) &\le U_1(\zeta)+\sum_{j=1}^{\infty}|U_{j+1}(\zeta)-U_j(\zeta)|\\ &\le W_1(\zeta)+\sum_{j=1}^{\infty}W_1(\zeta)(\gamma_0\varepsilon^{\frac{1}{3}})^{j}\\ &= W_1(\zeta)(1+O(\varepsilon^{\frac{1}{3}})).\end{aligned} \tag{4.34}$$

Since $\widetilde{U_1}(\zeta) = \lim_{n\to\infty}(U_n(\zeta)) = \lim_{n\to\infty}\sum_{j=0}^{n-1}[U_{j+1}(\zeta)-U_j(\zeta)]$, taking limits (as $j \to \infty$) on both sides of (4.30) shows that equation (2.14) has a solution $\widetilde{U_1}(\zeta)$ satisfying

$$\begin{aligned}\widetilde{U_1}(\zeta) =& W_1(\zeta)(1+O(\varepsilon^{\frac{1}{3}}))\\ =&(\sqrt{\zeta}K_1(\sqrt{\zeta})-P\sqrt{\zeta}I_1(\sqrt{\zeta}))(1+O(\varepsilon^{\frac{1}{3}})).\end{aligned} \tag{4.35}$$

Similar arguments show that equation (2.14) has another solution

$$\widetilde{U_2}(\zeta) = W_2(\zeta)(1+O(\varepsilon^{\frac{1}{3}})) = \frac{\sqrt{\zeta}I_1(\sqrt{\zeta})}{Q}(1+O(\varepsilon^{\frac{1}{3}})). \tag{4.36}$$

Since P and Q depend only on ε, and $w_1(\zeta)$ and $w_2(\zeta)$ are just linear combinations of $\widetilde{U_1}(\zeta)$ and $\widetilde{U_2}(\zeta)$, the result stated in the theorem is proved. □

Now, we are ready to construct the solution to B.V.P. (1.1) − (1.2). Coupling (2.10) and (2.13), we have

$$A(x) = \sqrt{\frac{\varepsilon}{2}}\left(\frac{1}{x}+\frac{x^2}{4\varepsilon^4}+\frac{1}{2\varepsilon}(1+2x^2)\right)^{-\frac{1}{4}}\exp\left\{-\frac{x^2}{4\varepsilon^3}\right\}\zeta^{-\frac{1}{4}}(x). \tag{4.37}$$

By Theorem 1, it is straightforward to show that B.V.P. (1.1) − (1.2) has two linearly independent solutions

$$y_1(x) = \sqrt{\frac{\varepsilon}{2}}\left(\frac{1}{x}+\frac{x^2}{4\varepsilon^4}+\frac{1}{2\varepsilon}(1+2x^2)\right)^{-\frac{1}{4}}\exp\left\{-\frac{x^2}{4\varepsilon^3}\right\}\zeta^{\frac{1}{4}}(x)K_1(\sqrt{\zeta})(1+O(\varepsilon^{\frac{1}{3}})), \tag{4.38}$$

$$y_2(x) = \sqrt{\frac{\varepsilon}{2}}\left(\frac{1}{x}+\frac{x^2}{4\varepsilon^4}+\frac{1}{2\varepsilon}(1+2x^2)\right)^{-\frac{1}{4}}\exp\left\{-\frac{x^2}{4\varepsilon^3}\right\}\zeta^{\frac{1}{4}}(x)I_1(\sqrt{\zeta})(1+O(\varepsilon^{\frac{1}{3}})). \tag{4.39}$$

In view of (4.7), (4.13), (4.14) and (4.15), we obtain

$$\lim_{x\to 0} y_1(x) = \frac{\varepsilon}{2}(1+O(\varepsilon^{\frac{1}{3}})), \qquad \lim_{x\to 0} y_2(x) = 0, \tag{4.40}$$

$$y_1(1) = \frac{\sqrt{\varepsilon\pi}}{2}\left(\frac{1}{4\varepsilon^4}\right)^{-\frac{1}{4}} \exp\left\{-\frac{1}{4\varepsilon^3}\right\} \exp\left\{-\sqrt{\zeta(1)}\right\}(1+O(\varepsilon^{\frac{1}{3}})). \tag{4.41}$$

The last estimate indicates that $y_1(1)$ is exponentially small as $\varepsilon \to 0$.

Let us define

$$Y_1(x) = \frac{2}{\varepsilon} y_1(x), \tag{4.42}$$

$$Y_2(x) = \frac{\sqrt{e}\, y_2(x)}{y_2(1)}. \tag{4.43}$$

The solution to the B.V.P. (1.1) $-$ (1.2) is given by

$$Y(x) = Y_1(x) + \left(1 - \frac{Y_1(1)}{Y_2(1)}\right) Y_2(x). \tag{4.44}$$

Note that $Y_1(1)/Y_2(1) = 2y_1(1)/(\varepsilon\sqrt{e})$ is exponentially small as ε tends to 0.

In summary, we have the following theorem.

Theorem 2. There exists a constant $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, the boundary-value problem (1.1) $-$ (1.2) has the unique solution

$$Y(x) = Y_1(x) + Y_2(x), \tag{4.45}$$

where $Y_1(x)$ and $Y_2(x)$ are given by (4.42) and (4.43), respectively.

5. **Simplification of our solution.**

To simplify our solution (4.45), we have to study the behavior of $\zeta^{1/2}(x)$.

Theorem 3. There is a positive number ρ_0 such that

$$\zeta^{\frac{1}{2}}(x) = 2\frac{\sqrt{x}}{\varepsilon}\left(1 + O(\frac{x^3}{\varepsilon^4}, \varepsilon^{\frac{1}{3}})\right) \qquad \text{for} \quad 0 \le x \le \rho_0 \varepsilon^{\frac{4}{3}} \tag{5.1}$$

and

$$\begin{aligned}\zeta^{\frac{1}{2}}(x) =& \frac{c_0}{\varepsilon^{\frac{1}{3}}} + \frac{x^2}{4\varepsilon^3} + \frac{1}{2}\log x - \frac{2}{3}\log\varepsilon + \frac{x^2}{2} \\ & - \frac{\varepsilon}{x}\left(1 + O(\frac{\varepsilon^4}{x^3}, \varepsilon^{\frac{1}{3}})\right) + O(\varepsilon^{\frac{1}{3}}\log\varepsilon) \qquad \text{for} \quad \rho_0\varepsilon^{\frac{4}{3}} \le x \le 1.\end{aligned} \tag{5.2}$$

Proof. By the definition of $\zeta^{1/2}(x)$ in (2.13), we have

$$\zeta^{\frac{1}{2}}(x) = \frac{1}{\varepsilon}\int_0^x \sqrt{\left(\frac{t^2}{4\varepsilon^4} + \frac{1}{t}\right)\left(1 + \frac{\frac{1}{2\varepsilon}(1+2t^2)}{\frac{t^2}{4\varepsilon^4} + \frac{1}{t}}\right)}\, dt. \tag{5.3}$$

It is easy to verify that

$$\frac{\frac{1}{2\varepsilon}(1+2t^2)}{\frac{t^2}{4\varepsilon^4} + \frac{1}{t}} = O(\varepsilon^{\frac{1}{3}}) \qquad \text{for} \quad 0 < t \le 1. \tag{5.4}$$

By the binomial expansion, we have

$$\begin{aligned}\zeta^{\frac{1}{2}}(x) = \varepsilon^{-\frac{1}{3}} \int_0^{x/\varepsilon^{4/3}} \sqrt{\frac{t^2}{4}+\frac{1}{t}}dt + \frac{1}{4}\int_0^{x/\varepsilon^{4/3}} \frac{1}{\sqrt{\frac{t^2}{4}+\frac{1}{t}}}dt\left[1+O(\varepsilon^{\frac{1}{3}})\right] \\ + \frac{1}{2\varepsilon^2}\int_0^x \frac{t^2}{\sqrt{\frac{t^2}{4\varepsilon^4}+\frac{1}{t}}}dt\left[1+O(\varepsilon^{\frac{1}{3}})\right].\end{aligned} \tag{5.5}$$

The first two integrals on the right-hand side of (5.5) can be evaluated by symbolic calculation (Mathematica 5.2), and the result is

$$\begin{aligned}\zeta^{\frac{1}{2}}(x) =& \frac{2\sqrt{x}}{\varepsilon}\ {}_2\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4}\right) + \frac{1}{3}\mathrm{arcsinh}\left(\frac{x^{\frac{3}{2}}}{2\varepsilon^2}\right)[1+O(\varepsilon^{\frac{1}{3}})] \\ &+ \frac{1}{2\varepsilon^2}\int_0^x \frac{t^2}{\sqrt{\frac{t^2}{4\varepsilon^4}+\frac{1}{t}}}dt[1+O(\varepsilon^{\frac{1}{3}})].\end{aligned} \tag{5.6}$$

This result can also be verified by using integral representations of the above two special functions. Now the theorem follows from (5.6) and the lemmas 4, 5 and 6. □

Lemma 4.

$$\frac{2\sqrt{x}}{\varepsilon}\ {}_2\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4}\right) = \frac{2\sqrt{x}}{\varepsilon}\left(1+O(\frac{x^3}{\varepsilon^4})\right) \qquad \textit{for} \quad 0 \le x \le \rho_0\varepsilon^{\frac{4}{3}} \tag{5.7}$$

and

$$\frac{2\sqrt{x}}{\varepsilon}\ {}_2\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4}\right) = \frac{c_0}{\varepsilon^{\frac{1}{3}}} + \frac{x^2}{4\varepsilon^3} - \frac{\varepsilon}{x}\left(1+O(\frac{\varepsilon^4}{x^3})\right) \qquad \textit{for} \quad \rho_0\varepsilon^{\frac{4}{3}} \le x \le 1, \tag{5.8}$$

where $c_0 = \frac{\Gamma(-\frac{2}{3})\Gamma(\frac{7}{6})2^{\frac{4}{3}}}{\Gamma(-\frac{1}{2})} \approx 2.64996$, and $\rho_0 = 2^{\frac{2}{3}}$.

Proof. We know that the hypergeometric series

$${}_2\mathrm{F}_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{5.9}$$

is absolutely convergent for $|z| \le 1$, provided that $\mathrm{Re}(c-a-b) > 0$. In particular, ${}_2\mathrm{F}_1(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4})$ is convergent for $|\frac{x^3}{4\varepsilon^4}| \le 1$ and (5.7) follows.

Referring to [1, p.559], we have the following connection formula

$$\begin{aligned}{}_2\mathrm{F}_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2\mathrm{F}_1(a,1-c+a;1-b+a;\frac{1}{z}) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2\mathrm{F}_1(b,1-c+b;1-a+b;\frac{1}{z}) \\ &\textit{for} \quad (|arg(-z)| < \pi).\end{aligned} \tag{5.10}$$

With $a = -\frac{1}{2}, b = \frac{1}{6}, c = \frac{7}{6}$ and $z = -\frac{x^3}{4\varepsilon^4}$, equation (5.10) becomes

$$\begin{aligned}{}_2\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4}\right) =& \frac{\Gamma(\frac{7}{6})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})\Gamma(\frac{10}{6})}\left(\frac{x^3}{4\varepsilon^4}\right)^{\frac{1}{2}}\ {}_2\mathrm{F}_1\left(-\frac{1}{2},-\frac{2}{3};\frac{1}{3};-\frac{4\varepsilon^4}{x^3}\right) \\ &+ \frac{\Gamma(\frac{7}{6})\Gamma(-\frac{2}{3})}{\Gamma(-\frac{1}{2})\Gamma(1)}\left(\frac{x^3}{4\varepsilon^4}\right)^{-\frac{1}{6}}\ {}_2\mathrm{F}_1\left(\frac{1}{6},0;\frac{5}{3};-\frac{4\varepsilon^4}{x^3}\right).\end{aligned} \tag{5.11}$$

Furthermore, it can be simplified to

$$ {}_2\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{6};\frac{7}{6};-\frac{x^3}{4\varepsilon^4}\right)=\frac{1}{8}\frac{x^{\frac{3}{2}}}{\varepsilon^2}\ {}_2\mathrm{F}_1\left(-\frac{1}{2},-\frac{2}{3};\frac{1}{3};-\frac{4\varepsilon^4}{x^3}\right)+c_0\frac{\varepsilon^{\frac{2}{3}}}{2x^{\frac{1}{2}}}, \tag{5.12}$$

where $c_0=\frac{\Gamma(\frac{7}{6})\Gamma(-\frac{2}{3})2^{\frac{4}{3}}}{\Gamma(-\frac{1}{2})}\approx 2.64996$. Multiplying both sides of (5.12) by $\frac{2\sqrt{x}}{\varepsilon}$ and making use of (5.9), we obtain (5.8). □

Lemma 5.

$$\frac{1}{3}\mathrm{arcsinh}\left(\frac{x^{\frac{3}{2}}}{2\varepsilon^2}\right)=\frac{x^{\frac{3}{2}}}{6\varepsilon^2}\left(1+O(\frac{x^3}{\varepsilon^4})\right)\qquad \textit{for}\quad 0\le x\le\rho_0\varepsilon^{\frac{4}{3}},$$

and

$$\frac{1}{3}\mathrm{arcsinh}\left(\frac{x^{\frac{3}{2}}}{2\varepsilon^2}\right)=\left(\frac{1}{2}\log x-\frac{2}{3}\log\varepsilon\right)+\frac{\varepsilon^4}{3x^3}\left(1+O(\frac{\varepsilon^4}{x^3})\right)\qquad \textit{for}\quad \rho_0\varepsilon^{\frac{4}{3}}\le x\le 1.$$

Lemma 6.

$$\frac{1}{2\varepsilon^2}\int_0^x\frac{t^2}{\sqrt{\frac{t^2}{4\varepsilon^4}+\frac{1}{t}}}dt=\frac{x^{\frac{7}{2}}}{7\varepsilon^2}\left(1+O(\frac{x^3}{\varepsilon^4})\right)\qquad \textit{for}\quad 0\le x\le\rho_0\varepsilon^{\frac{4}{3}},$$

and

$$\frac{1}{2\varepsilon^2}\int_0^x\frac{t^2}{\sqrt{\frac{t^2}{4\varepsilon^4}+\frac{1}{t}}}dt=\frac{x^2}{2}+O(\varepsilon^{\frac{8}{3}})\qquad \textit{for}\quad \rho_0\varepsilon^{\frac{4}{3}}\le x\le 1.$$

Lemmas 5 and 6 can be proved by using some basic calculus and the asymptotic formulas of arcsinh z [1, p.88].

From Theorem 3 and the asymptotic formulas of $K_1(z)$ and $I_1(z)$, the solution $Y(x)$ in (4.45) can be simplified to

$$Y(x)=\begin{cases}\frac{2\sqrt{x}}{\varepsilon}K_1\left\{\frac{2\sqrt{x}}{\varepsilon}(1+o(1))\right\}\exp\left\{-\frac{x^2}{4\varepsilon^3}\right\}(1+o(1)) & \textit{for } 0<x\ll O(\varepsilon^{\frac{4}{3}}),\\ \exp\left\{\frac{x^2}{2}-\frac{\varepsilon}{x}(1+o(1))\right\}(1+o(1)) & \textit{for } O(\varepsilon^{\frac{4}{3}})\ll x\le 1\ .\end{cases} \tag{5.13}$$

We claim that, for $O(\varepsilon^2)\ll x\ll O(\varepsilon)$, the solution $Y(x)$ is exponentially small. To see this, we note from (5.13) that the terms $K_1\left(\frac{2\sqrt{x}}{\varepsilon}\right)\exp\left(-\frac{x^2}{4\varepsilon^3}\right)$ for $O(\varepsilon^2)\ll x\le O(\varepsilon^{4/3})$ and $\exp\left(\frac{x^2}{2}-\frac{\varepsilon}{x}\right)$ for $O(\varepsilon^{4/3})\le x\ll O(\varepsilon)$ are both exponentially small.

To compare our solution with Bender and Orszag's solution, we evaluate the solution at some particular points. For example, at $x=\varepsilon^{41/28}$, Bender and Orszag's solution (1.3) gives

$$y_{\mathrm{unif}}(\varepsilon^{41/28})=\frac{2\sqrt{\varepsilon^{41/28}}}{\varepsilon}K_1\left(\frac{2\sqrt{\varepsilon^{41/28}}}{\varepsilon}\right)+e^{-\frac{\varepsilon}{\varepsilon^{41/28}}}+e^{\frac{(\varepsilon^{41/28})^2}{2}}-1\sim\frac{\varepsilon^{41/14}}{2},$$

while our solution yields

$$\begin{aligned}Y(\varepsilon^{41/28})&\sim\sqrt{\frac{\pi}{\varepsilon}}(\varepsilon^{41/28})^{\frac{1}{4}}\exp\left\{-\frac{(\varepsilon^{41/28})^2}{4\varepsilon^3}-\frac{2\sqrt{\varepsilon^{41/28}}}{\varepsilon}\right\}\\&\sim\frac{\sqrt{\pi}}{\varepsilon^{15/112}}\exp\{-\frac{2}{\varepsilon^{15/56}}-\frac{1}{4\varepsilon^{1/14}}\}.\end{aligned}$$

Moreover, at $x = \varepsilon^{7/6}$, Bender and Orszag's solution (1.3) gives

$$y_{\text{unif}}(\varepsilon^{7/6}) = \frac{2\sqrt{\varepsilon^{7/6}}}{\varepsilon} K_1\left(\frac{2\sqrt{\varepsilon^{7/6}}}{\varepsilon}\right) + e^{-\frac{\varepsilon}{\varepsilon^{7/6}}} + e^{\frac{(\varepsilon^{7/6})^2}{2}} - 1 \sim \frac{\varepsilon^{7/3}}{2},$$

while our solution gives

$$Y(\varepsilon^{7/6}) \sim \exp\left\{\frac{\varepsilon^{(7/6)^2}}{2} - \frac{\varepsilon}{\varepsilon^{7/6}}(1 + o(1))\right\} \sim \exp\{-\frac{1}{\varepsilon^{1/6}}\}.$$

In conclusion, for $O(\varepsilon^2) \ll x \ll O(\varepsilon)$ Bender and Orszag's asymptotic solution (1.3) seems to be incorrect since it shows that the leading term in the asymptotic approximation of the exact solution is of algebraic order, whereas our solution (4.45) shows that the true solution decays exponentially; this phenomenon has never been mentioned in the existing literature.

6. **Triple-deck problem.** Based on the experience from studying problem (1.1) − (1.2), we believe that our method also works for problem (1.7). In fact, problem (1.7) is much simpler than problem (1.1) − (1.2) due to the lack of singularity at the origin. Since the techniques are the same, we present only a brief outline of the derivation of an asymptotic solution to the triple-deck problem (1.7).

In parallel to Lemma 1, we have

Lemma 7. Under the transformations

$$\zeta(x) = \frac{1}{2\varepsilon^3}\int_0^x \sqrt{t^6 + 4\varepsilon^4 + \varepsilon^3(6t^2 - 4t^3)}dt \tag{6.1}$$

and

$$y(x) = (\zeta'(x))^{-\frac{1}{2}} \exp\left\{-\frac{x^4}{8\varepsilon^3}\right\} U(\zeta(x)), \tag{6.2}$$

(1.7) is converted to

$$\frac{d^2U}{d\zeta^2} - U = \phi(\zeta)U, \tag{6.3}$$

where

$$\phi(\zeta(x)) = \frac{1}{2}\frac{\zeta'''(x)}{(\zeta'(x))^3} - \frac{3}{4}\frac{(\zeta''(x))^2}{(\zeta'(x))^4}. \tag{6.4}$$

The estimate $\phi(\zeta) = O(\varepsilon^{2/3})$ implies that

$$\frac{d^2U}{d\zeta^2} - U = 0 \tag{6.5}$$

can be used as a perturbed equation to (6.3), and guarantees success of the successive approximation method used in Sect. 4. As before, one can show that equation (1.7) has two linearly independent solutions

$$\begin{aligned} y_1(x) =& \sqrt{2\varepsilon^3}\left(x^6 + 4\varepsilon^4 + \varepsilon^3(6x^2 - 4x^3)\right)^{-\frac{1}{4}} \exp\left\{-\frac{x^4}{8\varepsilon^3}\right\} \\ &\times \exp\left\{-\frac{1}{2\varepsilon^3}\int_0^x \sqrt{t^6 + 4\varepsilon^4 + \varepsilon^3(6t^2 - 4t^3)}dt\right\}(1 + O(\varepsilon^{\frac{1}{3}})) \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} y_2(x) =&\sqrt{2\varepsilon^3}\left(x^6+4\varepsilon^4+\varepsilon^3(6x^2-4x^3)\right)^{-\frac{1}{4}}\exp\left\{-\frac{x^4}{8\varepsilon^3}\right\} \\ &\times\exp\left\{+\frac{1}{2\varepsilon^3}\int_0^x\sqrt{t^6+4\varepsilon^4+\varepsilon^3(6t^2-4t^3)}dt\right\}(1+O(\varepsilon^{\frac{1}{3}})). \end{aligned} \tag{6.7}$$

The solution which satisfies the boundary conditions of (1.7) is given by

$$Y(x)=\frac{\alpha}{\sqrt{\varepsilon}}y_1(x)+\frac{\beta}{y_2(1)}y_2(x), \tag{6.8}$$

where $y_1(x)$ and $y_2(x)$ are given in (6.6) and (6.7), respectively.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables," Dover, New York, 1992.

[2] C. M. Bender and S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers," McGraw-Hill, New York, 1978.

[3] R. E. Langer, *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point*, Trans. Amer. Math. Soc., **67** (1949), 461–490.

[4] A. H. Nayfeh, "Introduction to Perturbation Techniques," John Wiley and Sons, New York, 1981.

[5] A. H. Nayfeh, "Problems in Perturbation," John Wiley and Sons, New York, 1985.

[6] F. W. J. Olver, "Asymptotics and Special Functions," Academic Press, New York, 1974.

[7] R. Wong, Five Lectures on Asymptotics, in "Differential Equations & Asymptotic Theory in Mathematical Physics" (Eds. Chen Hua and R. Wong), World Scientific, Singapore, 2004, 189–262.

[8] R. Wong and H. Yang, *On an internal layer problem*, J. Comput. Appl. Math., **144** (2002), 301–323.

[9] R. Wong and H. Yang, *On a boundary-layer problem*, Stud. Appl. Math., **108** (2002), 369–398.

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